Inequalities for Time Reversible Chains

Application areas: Birth and death processes; repairable systems in reliability; random walks on graphs; queuing networks; Markov chain Monte Carlo.

$\{X(t), t \ge 0\}$, irreducible finite state chain. Infinitesimal matrix Q.

Time reversible: $\pi_i q_{ij} = \pi_j q_{ji}$

$$D_\pi = \begin{pmatrix} \pi_1 & & 0 \\ & \ddots & \\ 0 & & \pi_n \end{pmatrix}$$

Time reversible equivalent to,

$$D_\pi Q = Q' D_\pi$$

mult by $D_\pi^{-1/2}$ on left and right

then, $M \overset{def}{=} D_\pi^{1/2} Q D_\pi^{-1/2} = D_\pi^{-1/2} Q' D_\pi^{1/2} = M'$

Q is similar to M, a real symmetric matrix. Thus the eigenvalues of Q are real.

The eigenvectors for M can be chosen to be orthonormal.

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Eigenvalues of $-Q$, $-M$

$$\lambda_0 = 0 < \lambda_1 \le \lambda_2 \cdots \le \lambda_n$$

$\uparrow$ MULTIPLICITY 1

eigenvectors for $-M$

$t_0, t_1 \cdots t_n$ orthonormal

$-M$: eigenvector

$$t_0 = \sqrt{\pi} = D_\pi^{1/2}\mathbf{1} = \begin{pmatrix} \sqrt{\pi_0} \\ \vdots \\ \sqrt{\pi_n} \end{pmatrix}$$

$$\tau = \frac{1}{\lambda_1} \text{ relaxation time}$$

Justification of relaxation time <u>designation</u>

e^{2At} eigenv. $1, e^{-2\lambda_1 t}, \ldots e^{-2\lambda_n t}$

$$(-M) = \Pi_D^{1/2}(-Q)\,\Pi_D^{-1/2}$$

$$e^{Mt} = \Pi_D^{1/2} e^{Qt}\, \Pi_D^{-1/2}$$

$$= \Pi_D^{1/2} P(t)\, \Pi_D^{-1/2}$$

$$\sup_{(x, t_0) = 0} \frac{\|e^{Mt}x\|^2}{\|x\|^2} = e^{-2\lambda_1 t}$$

$$\Rightarrow \|e^{Mt}x\|^2 \le \|x\|^2 e^{-2\lambda_1 t}$$

(3)

Consider a probabity distribution v on the state space

Let $x = D_\pi^{-1/2}(v - \pi)$ so that

$$x_i = \frac{v_i - \pi_i}{\sqrt{\pi_i}}$$

Then $\|x\|^2 = \sum \frac{(v_i - \pi_i)^2}{\pi_i} = \chi_v^2$

and $(e^{Mt}x)_i = \frac{P_v(X(t)=i) - \pi(i)}{\sqrt{\pi_i}}$

$$= \frac{v_t(i) - \pi(i)}{\sqrt{\pi_i}}$$

$\therefore \|e^{Mt}x\|^2 = \sum \frac{(v_t(i) - \pi(i))^2}{\pi_i}$

$$= \chi_v^2(t)$$

$\therefore \chi_v^2(t) = \|e^{Mt}x\|^2 \leq \chi_v^2(0)e^{-2\lambda_1 t}$

$$TV_v(X(t),\pi) = \max_A |P_v(X(t)\in A) - \pi(A)|$$

$$= \frac{1}{2}\sum |P_v(X(t)=i) - \pi(i)|$$

$$= \frac{1}{2} E_\pi \left| \frac{P_v(X(t)) - \pi}{\pi} \right|$$

$$\le \frac{1}{2}\sqrt{E_\pi \left(\frac{P_v(X(t)) - \pi}{\pi}\right)^2}$$

$$= \frac{1}{2}\sqrt{\chi^2_v(t)}$$

$$\le \frac{1}{2}\sqrt{\chi^2_v(0)}\, e^{-\lambda_1 t}$$

For $t = \frac{m}{\lambda_1}$ $TV_v(X(t),\pi) \le \frac{1}{2}\sqrt{\chi^2_v(0)}\, e^{-m}$

Next given g - function on state space

$$\mu_g = E_\pi g(X_0)$$

$$\sigma_g^2 = Var_\pi g(X_0)$$

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Let $x = D_\pi^{1/2}(g - \mu_g)$

$$x_i = \sqrt{\pi_i}(g_i - \mu_g)$$

Then $(x, \phi_0) = \sum \pi_i (g_i - \mu_g) = 0$

and $(e^{Mt} x)_i = \sqrt{\pi_i}(E_i g(X_t) - \mu_g)$

Then,

$$\sum \pi_i (E_i(g(X_t)) - \mu_g)^2$$

$$= \|M_t x\|^2 \le \|x\|^2 e^{-2\lambda_1 t}$$

$$= \sigma_g^2 e^{-2\lambda_1 t}$$

$\therefore$ ~~$\mathrm{Var}_\pi E(g(X_t) | X(0))$~~

$\therefore \mathrm{Var}_\pi E(g(X_t)) | X(0))$

$$\le \sigma_g^2 e^{-2\lambda_1 t}$$

Furthermore:

$$|E_\nu g(X_t) - E_\pi g(X_0)| \leq \sigma_g \sqrt{\chi^2_\nu(0)}\, e^{-\lambda_1 t}$$

as a variation of the above argument.

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$$\sup_t |\mathrm{Corr}(h(X_0), g(X_t))| = e^{-\lambda_1 t}$$

<u>First passage times</u>

$$A \subset S,\ T_A$$

$-\tilde{Q}$ eigen. $0 < \gamma_1 \le \gamma_2 \cdots \le \gamma_n$

$$T_A - t \mid T_A > t$$

$$\to \mathrm{Expo}(\gamma_1)$$

$\mathrm{Expo}(\gamma_1)$ quasi-stationary first passage time distribution to A

Principle: "If $\frac{\gamma_1}{\lambda_1} = \frac{1}{E_\alpha T_A}$ is small then

$$\mathcal{L}_\pi T_A \approx \mathrm{Expo}(E_\pi T_A) \approx \mathrm{Expo}(\tfrac{1}{\gamma_1})$$

and $E_i T_A$ is close to $E_\pi T_A$ and $\frac{1}{\gamma_1}$ for most i."

$\gamma_1 \quad \gamma_2 \quad \cdots \quad \gamma_n$

$\varphi_1 \quad \varphi_2 \quad \cdots \quad \varphi_n$

$$P_\pi(T_A > t) = \sum_1^n P_v e^{-\gamma_v t}$$

$P_v > 0, \ \sum P_v = 1 - \pi(A)$

Completely monotone.

$$\rho = \frac{\mu_2}{2\mu^2} - 1 \qquad \mu_2 = E_\pi T_A^2$$
$$\mu = E_\pi T_A$$

$$d(T_A) = \sup_t \left| P_\pi(T_A > t) - e^{-t/E_\pi T_A} \right|$$
$$\leq \frac{\rho}{\rho+1} = 1 - \frac{2\mu^2}{\mu_2} \quad \underline{\text{sharp}}$$

want to express bound in terms of $\frac{\gamma_1}{\lambda_1}$

$$d(T_A) \leq \frac{\gamma_1}{\lambda_1}$$

also $d(T_A) \leq P_\pi\left(T_A \leq \frac{1}{\lambda_1}\right)$

Also:

$$0 \leq e^{-\gamma_1 t} - P_\pi(T_A > t) \leq \frac{\gamma_1}{\lambda_1} e^{-\gamma_1 t}$$

$$\frac{1}{\gamma_1}-\frac{1}{\lambda_1} \le E_\pi T_A \le \frac{1}{\gamma_1}$$

Other Methodology

Interlacing Eigenvalues -

Isomorphism with star chain

Dirichlet Forms

Results 1) α quasi-stationary distribution on state space

$$TV(\alpha,\pi) \le \frac{1}{2}\sqrt{\frac{\gamma_1}{\lambda_1-\gamma_1}}$$

2) $$TV_\pi((X(t)\,|\,T_A>t),\alpha) \le \frac{1}{2}\left(1-\frac{\gamma_1}{\lambda_1}\right)^{-3/2}\left(\frac{\gamma_1}{\lambda_1}-\pi(A)\right)^{1/2} \cdot (1-\pi(A))^{1/2} e^{-(\lambda_1-\gamma_1)t}$$

3) $$\sup_s \frac{P_w(T_A(t)>s)-P_\pi(T_A>s)}{P_\pi(T_A>s)} \le \frac{1}{2}\sqrt{\chi^2_w}\left(1-\frac{\gamma_1}{\lambda_1}\right)^{-1}e^{-\lambda_1 t}$$

4) $P_w(T_A > t) \le \frac{1}{2}\left[w(A^c) + \sqrt{\pi(A^c)\sum \frac{w_i^2}{\pi_i}}\right]e^{-\gamma_1 t}$

5) $|P_w(T_A > s) - P_\pi(T_A > s)| \le \frac{1}{2}\sqrt{\chi_w^2}\, e^{-\gamma_1 s}$

6) $\frac{1}{\gamma_1^2}\left(\frac{\gamma_1^2}{\lambda_1^2}\right) \le \frac{1}{\gamma_1^2} - \mathrm{Var}_\pi T_A \le \frac{\pi^2(A)}{\gamma_1^2}$

Repairable System

mod 1	mod 2	
parallel	2 out of 3	Series system
2 comp.	$\lambda = 1.3, \mu = 25$	of modules
$\lambda = 1, \mu = 20$		

get bounds using only γ_1, λ_1

$$\frac{\gamma_1}{\lambda_1} \le .01823$$

$$2.39976 \le E_\pi T_A \le 2.40130$$

$$2.42354 \le SD_\pi T_A \le 2.42384$$

$$2.00025 \le \mathrm{Skew}_\pi T_A \le 2.00100$$

$$6.01104 \le \mathrm{Kur}_\pi T_A \le 6.00399$$

7) $\mathcal{L}_\pi T_A \sim \sum_1^m Y_i, \; P_r(Y_i > t) = \left(1 - \frac{\gamma_i}{\lambda_i}\right)e^{-\gamma_i t}$

Technical result - Recently Fill found a sample path interpretation